

Gauge Theory for Quantum Spin Glasses

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The gauge theory for random spin systems is extended to quantum spin glasses to derive a number of exact and/or rigorous results. The transverse Ising model and the quantum gauge glass are shown to be gauge invariant. For these models, an identity is proved that the expectation value of the gauge invariant operator in the ferromagnetic limit is equal to the one in the classical equilibrium state on the Nishimori line. As a result, a set of inequalities for the correlation function are proved, which restrict the location of the ordered phase. It is also proved that there is no long-range order in the two-dimensional quantum gauge glass in the ground state. The phase diagram for the quantum *XY* Mattis model is determined.

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1. Introduction

The problem of spin glasses has been attracting continued attention.^{1,2)} After the pioneering work by Edwards and Anderson (EA),³⁾ Sherrington and Kirkpatrick have exactly solved the mean-field model by assuming the replica symmetry.⁴⁾ Parisi has proposed the replica symmetry breaking solution and established the theoretical picture that the low-temperature phase is composed of infinitely many stable states with ultrametric structure.⁵⁾

A topic of active investigations in recent years concerns the properties of short-range systems. Numerical studies have provided strong evidence for the existence of the spin glass (SG) transition for both the EA model⁶⁻⁹⁾ and the *XY* gauge glass¹⁰⁻¹³⁾ in three dimensions but against it for the two-dimensional EA model.^{8,9)} For the two-dimensional gauge glass, although the long-range SG order has been denied rigorously,¹⁴⁾ it is still possible that the system has a quasi long-range order in which the SG correlation decays in a power law. There remains the controversy about the existence of this order: some numerical studies have supported the absent of a finite-temperature transition^{13,15,16)} but some groups argue against such a conclusion.^{17,18)}

Analytical calculations for spin glasses in finite dimensions are difficult because of randomness and frustration. However, a method using the gauge symmetry of the system is well-known as a powerful technique.^{19,20)} This method provides various rigorous results, for instance, the exact internal energy and an upper bound for the specific heat in the special region of the phase diagram. Another noteworthy result is a set of inequalities for the correlation function, which restrict the topology of the phase diagram. In this relation, it has been suggested that the phase boundary between the ferromagnetic (FM) and SG phases is vertical by modifying the probability distribution.²¹⁾ These results are generalized to a wider class of systems including the usual Ising SG and the *Z_q* and *XY* gauge glasses.²²⁾

Although the gauge theory provides us with surprising results, its targets have so far been limited to classical spin systems. In the present paper, we generalize this theory so that it applies to quantum spin systems. A difficulty of this generalization is the fact that we must define the gauge

transformation for spins without violating the commutation rule. We circumvent this problem by using a rotational operator on the Hilbert space as the gauge transformation.

This paper consists of six sections. In the next section, we formulate the gauge transformation in two quantum spin glasses, the transverse Ising model and the quantum gauge glass (QGG), and show that these models have gauge symmetry. In §3, we prove an identity for a gauge invariant operator. This identity is valid even when the system parameters depend on time following the time-dependent Schrödinger equation. In §4, we derive a set of inequalities for correlation functions and order parameters. These results restrict the location of the FM phase or the Kosterlitz–Thouless (KT) phase in the phase diagram. In §5, we extend these inequalities to the ground state. The resulting inequalities for the order parameters show that the FM order does not exist at zero-temperature in the two-dimensional QGG. In §6, we consider the quantum *XY* Mattis model and determine its phase diagram. The last section is devoted to summary.

2. Gauge Transformation for Random Quantum Spin Systems

2.1 Transverse Ising model

First, let us consider the random-bond Ising model in a transverse field. The Hamiltonian for this model is written as

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x, \quad (1)$$

where σ_i^α is the Pauli matrix at site i . Although we treat spin-1/2 systems in this paper, one can straightforwardly generalize all the results to spin- S systems. There is no restriction in the spatial dimensionality or lattice structure. The exchange interaction J_{ij} is a quenched random variable. One of the useful probability distributions for J_{ij} is the binary distribution

$$P(J_{ij}) = p \delta(J_{ij} - J) + (1 - p) \delta(J_{ij} + J). \quad (2)$$

It is convenient for later arguments to rewrite this distribution as

$$P(J_{ij}) = \frac{e^{K_p \tau_{ij}}}{2 \cosh K_p}, \quad K_p = \frac{1}{2} \log \frac{p}{1-p}, \quad (3)$$

where $\tau_{ij} = J_{ij}/J$ is the sign of the exchange interaction J_{ij} . Another useful distribution is the Gaussian distribution

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} \exp\left(-\frac{(J_{ij} - J_0)^2}{2J^2}\right), \quad (4)$$

where J_0 and J^2 denote the average and variance, respectively.

For quantum spin systems, the classical gauge transformation, which simultaneously changes the sign of all components, is not valid because the commutation rule $[\sigma_i^x, \sigma_i^y] = 2i\sigma_i^z$ is changed to $[\sigma_i^x, \sigma_i^y] = -2i\sigma_i^z$. Thus we define a gauge transformation for spins using a unitary operator as

$$U : \sigma_i^\alpha \rightarrow G\sigma_i^\alpha G^{-1}, \quad G = \prod_i G_i, \quad (5)$$

$$G_i = \begin{cases} 1_i & (\xi_i = +1) \\ \exp\left(-\frac{i\pi}{2}\sigma_i^x\right) & (\xi_i = -1) \end{cases},$$

where ξ_i is a classical gauge variable at site i and takes two values ± 1 . If $\xi_i = -1$, $\sigma_i^{y,z} \rightarrow -\sigma_i^{y,z}$ and $\sigma_i^x \rightarrow \sigma_i^x$. Equivalently we can write

$$U : (\sigma_i^x, \sigma_i^y, \sigma_i^z) \rightarrow (\sigma_i^x, \xi_i\sigma_i^y, \xi_i\sigma_i^z). \quad (6)$$

A difference of gauge transformations between quantum and classical systems is the transformation rule of σ_i^x .

The gauge transformation for the bond variables $\{J_{ij}\}$ is the same as in classical systems, namely

$$V : J_{ij} \rightarrow J_{ij}\xi_i\xi_j. \quad (7)$$

The transverse-field term in eq. (1) does not change by the gauge transformation.

The Hamiltonian (1) is clearly invariant under the successive operations of V and U : $(UV)H = H$. However, the distribution function of bond configuration is changed, for the $\pm J$ Ising model, as

$$P(J_{ij}) \rightarrow \frac{e^{K_p \tau_{ij} \xi_i \xi_j}}{2 \cosh K_p}. \quad (8)$$

It is important for the following argument that this transformed distribution is proportional to the Boltzmann factor of a classical system. Similarly, the Gaussian distribution (4) is changed as

$$P(J_{ij}) \rightarrow \frac{1}{\sqrt{2\pi J^2}} \exp\left(-\frac{J_{ij}^2 + J_0^2}{2J^2}\right) \exp\left(\frac{J_0}{J^2} J_{ij} \xi_i \xi_j\right). \quad (9)$$

To simplify the arguments, we focus on the binary distribution (3) and (8), hereafter. It is straightforward to apply the same methods to the Gaussian distribution.

2.2 Quantum gauge glass

Next, we consider the quantum gauge glass (QGG). Similarly to the transverse Ising model, the gauge transformation is defined by the rotation operator.

To properly define the quantum version of gauge glass, let us first consider the Hamiltonian of the classical gauge glass (CGG),

$$H_{cl} = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j - \omega_{ij}). \quad (10)$$

This Hamiltonian can be rewritten using the spin vector

composed of x and y components $\mathbf{S}_i = \begin{pmatrix} S_i^x \\ S_i^y \end{pmatrix}$ and rotational matrix in the XY plane, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ as

$$H_{cl} = -J \sum_{\langle ij \rangle} {}^t \mathbf{S}_i R(\omega_{ij}) \mathbf{S}_j. \quad (11)$$

Thus this model can be quantized straightforwardly by replacing the elements of the above spin vectors by the Pauli matrices. The Hamiltonian of the QGG is therefore written explicitly as

$$H = -J \sum_{\langle ij \rangle} \left\{ \cos \omega_{ij} \left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) - \sin \omega_{ij} \left(\sigma_i^x \sigma_j^y - \sigma_i^y \sigma_j^x \right) \right\}. \quad (12)$$

The phase factor $\omega_{ij} \in [0, 2\pi)$ is a quenched random variable whose probability distribution is of cosine type

$$P(\omega_{ij}) = \frac{e^{K_p \cos \omega_{ij}}}{2\pi I_0(K_p)}, \quad (13)$$

a periodic Gaussian (Villain) type

$$P(\omega_{ij}) = \sqrt{\frac{K_p}{2\pi}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{K_p(\omega_{ij} - 2\pi n)^2}{2}\right) \quad (14)$$

or a binary type

$$P(\omega_{ij}) = p \delta(\omega_{ij}) + (1-p) \delta(\omega_{ij} - \pi). \quad (15)$$

Equation (12) is a special case of the XY model with Dzyaloshinskii–Moriya interactions. This Hamiltonian is written as

$$H = -J \sum_{\langle ij \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - \sum_{\langle ij \rangle} J_{ij} (\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j)_z, \quad (16)$$

where the second term is the random Dzyaloshinskii–Moriya interaction. If we set new parameters,

$$\tilde{J}_{ij} = \sqrt{J^2 + J_{ij}^2}, \quad \omega_{ij} = -\tan^{-1}\left(\frac{J_{ij}}{J}\right), \quad (17)$$

the above Hamiltonian is rewritten as

$$H = - \sum_{\langle ij \rangle} \tilde{J}_{ij} \left\{ \cos \omega_{ij} \left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) - \sin \omega_{ij} \left(\sigma_i^x \sigma_j^y - \sigma_i^y \sigma_j^x \right) \right\}. \quad (18)$$

This is equal to eq. (12) except that the interaction depends on bonds.

In the CGG, the gauge transformation for spins is defined by the shift of spin variables as $\phi_i \rightarrow \phi_i - \psi_i$, where $\psi_i \in [0, 2\pi)$ is the gauge variable. Using the same notation as in eq. (11), this transformation is expressed as

$$U : \mathbf{S}_i \rightarrow R(-\psi_i) \mathbf{S}_i. \quad (19)$$

Thus we use this definition of gauge transformation for the QGG. Using a rotational operator on the Hilbert space, we define

$$U : \sigma_i \rightarrow G\sigma_i G^{-1} \quad G = \prod_i \exp\left(-\frac{i\psi_i}{2}\sigma_i^z\right). \quad (20)$$

The transformation rule for the transposed vector ${}^t\sigma_i$ is defined as

$$U : {}^t\sigma_i \rightarrow {}^t\sigma_i R(\psi_i) = G {}^t\sigma_i G^{-1}. \quad (21)$$

The gauge transformation of random variables is the same as in the classical case,

$$V : \omega_{ij} \rightarrow \omega_{ij} - \psi_i + \psi_j. \quad (22)$$

Under the gauge transformation UV , the Hamiltonian is invariant because

$$(UV)H = -J \sum_{(ij)} \sigma_i R(\psi_i) R(\omega_{ij} - \psi_i + \psi_j) R(-\psi_j) \sigma_j = H, \quad (23)$$

where we used the property of rotation matrices, $R(\psi)R(\phi) = R(\psi + \phi)$. The probability distribution (13) is changed as

$$P(\omega_{ij}) \rightarrow \frac{e^{K_p \cos(\omega_{ij} - \psi_i + \psi_j)}}{2\pi I_0(K_p)}. \quad (24)$$

This transformed distribution is proportional to the Boltzmann factor for the CGG. If we choose the Gaussian or binary type, the Boltzmann factor for the Villain or $\pm J$ model appears, respectively.

3. Identity for Gauge Invariant Operators

The gauge symmetry of the Hamiltonian yields a useful identity for gauge invariant operators. First, let us suppose that the system was initially in the perfect FM state $|F_z\rangle$ in the transverse Ising model. This state appears in the FM limit, $T = 0$, $p = 1$, $h = 0$. The gauge transformation operator G defined in eq. (5) changes this state as

$$G|F_z\rangle = |\xi\rangle, \quad |\xi\rangle = |\xi\rangle_1 |\xi\rangle_2 \cdots |\xi\rangle_N. \quad (25)$$

If $\xi_i = +1$, $|\xi\rangle_i$ denotes the state with up spin in the z direction, and if $\xi_i = -1$, the spin at site i is down.

Using the property of $|F_z\rangle$ in (25), we prove the following identity for a gauge-invariant operator Q which satisfies $Q = (UV)Q$ or equivalently $VQ = G^{-1}QG$,

$$[\langle Q \rangle_{F_z}] = [\langle Q \rangle_{\rho_{\text{cl}}(K_p)}], \quad (26)$$

where $\langle \cdot \rangle_{\rho_{\text{cl}}(K_p)}$ is the expectation value for the classical equilibrium state on the Nishimori line (NL), that is,

$$\langle Q \rangle_{\rho_{\text{cl}}(K_p)} = \text{Tr} \rho_{\text{cl}}(K_p) Q, \quad (27)$$

$$\rho_{\text{cl}}(K_p) = \frac{e^{K_p \sum_{ij} \tau_{ij} \sigma_i^z \sigma_j^z}}{\text{Tr} e^{K_p \sum_{ij} \tau_{ij} \sigma_i^z \sigma_j^z}}.$$

To prove the identity (26), we apply the gauge transformation for the configuration of randomness of eq. (7) appearing on left-hand side of eq. (26). This operation does not change its value because the transformation V of eq. (7) only changes the order of the summation over τ_{ij} . Thus the left-hand side of eq. (26) is rewritten as

$$[\langle Q \rangle_{F_z}] = \sum_{\{\tau_{ij}\}} \frac{e^{K_p \sum \tau_{ij} \xi_i \xi_j}}{(2 \cosh K_p)^{N_B}} \langle F_z | VQ | F_z \rangle = \sum_{\{\tau_{ij}\}} \frac{e^{K_p \sum \tau_{ij} \xi_i \xi_j}}{(2 \cosh K_p)^{N_B}} \langle \xi | Q | \xi \rangle, \quad (28)$$

where we used the assumption that the operator Q is gauge invariant, $VQ = G^{-1}QG$. Since the expectation value on the left-hand side does not depend on ξ , the summation over ξ and division by 2^N yield

$$[\langle Q \rangle_{F_z}] = \sum_{\{\tau_{ij}\}} \frac{1}{2^N (2 \cosh K_p)^{N_B}} \times \sum_{\{\xi\}} e^{K_p \sum \tau_{ij} \xi_i \xi_j} \langle \xi | Q | \xi \rangle. \quad (29)$$

The last part of the right-hand side is rewritten in terms of $\rho_{\text{cl}}(K_p)$ as

$$\sum_{\{\xi\}} e^{K_p \sum \tau_{ij} \xi_i \xi_j} \langle \xi | Q | \xi \rangle = \left(\sum_{\{\xi\}} e^{K_p \sum \tau_{ij} \xi_i \xi_j} \right) (\text{Tr} \rho_{\text{cl}}(K_p) Q). \quad (30)$$

Therefore, we obtain

$$[\langle Q \rangle_{F_z}] = \sum_{\{\tau_{ij}\}} \frac{\sum_{\xi} e^{K_p \sum \tau_{ij} \xi_i \xi_j}}{2^N (2 \cosh K_p)^{N_B}} \text{Tr} \rho_{\text{cl}}(K_p) Q. \quad (31)$$

Since $\text{Tr} \rho_{\text{cl}}(K_p) Q$ is invariant under the transformation V , this is identical to the right-hand side of eq. (26)

The above result can be generalized to the case that the transverse field $h(t)$ depends on time, following the classical example.²³⁾ We consider the zero-temperature time evolution following the Schrödinger equation. Using the time ordered product, the time evolution operator is written as

$$U_t = T \exp \left(-i \int_0^t H(t') dt' \right). \quad (32)$$

Since the time dependence of the Hamiltonian does not invalidate the gauge symmetry, this operator is also gauge invariant

$$(UV)U_t = G(VU_t)G^{-1} = U_t. \quad (33)$$

Examples of the gauge invariant operator include the transverse magnetization $\sigma_i^x(t) = U_t^\dagger \sigma_i^x U_t$, the autocorrelation function $\sigma_i^z(0) \sigma_i^z(t)$ and the interaction term $H_0(t)$ of the Hamiltonian (the first term on the right-hand side of eq. (1)).

For the QGG, we can prove a similar identity

$$[\langle Q \rangle_{F_x}] = [\langle Q \rangle_{\rho_{\text{cl}}(K_p)}]. \quad (34)$$

Here one should note that $\rho_{\text{cl}}(K_p)$ is different from the normal density operator. If we choose the cosine-type distribution (13), $\rho_{\text{cl}}(K_p)$ is defined by the Boltzmann factor for the CGG as

$$\rho_{\text{cl}}(K_p) = \frac{\text{Tr}_\psi e^{K_p \sum_{ij} \cos(\omega_{ij} - \psi_i + \psi_j)} |\psi\rangle \langle \psi|}{\text{Tr}_\psi e^{K_p \sum_{ij} \cos(\omega_{ij} - \psi_i + \psi_j)}}, \quad (35)$$

where $|\psi\rangle = G|F_x\rangle$ and Tr_ψ stands for integration over ψ_i from 0 to 2π . Since the state vector $|\psi\rangle$ does not diagonalize the Hamiltonian for the quantum gauge glass, $\rho_{\text{cl}}(K_p)$ is not equal to the density operator

$$\rho(K_p) = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}. \quad (36)$$

The identities (26) and (34) show that the expectation value of gauge invariant operator in the FM limit is equal to the one in the classical equilibrium state on the NL. The equivalence of the two states has already been proved in the dynamical gauge theory for classical systems.^{23,24)} The

present results are generalization of these dynamical cases to quantum systems. Note that the zero-temperature time evolution for quantum systems is deterministic in contrast to the stochastic dynamics for the classical SG.

4. Correlation Function and Order Parameter

Using the identities proved in the previous section, we can derive a class of inequalities for the correlation function. First, we treat the transverse Ising model. Since the correlation function is not invariant under the gauge transformation, let us consider the following gauge-invariant quantity

$$Q = \sigma_i^z \sigma_j^z \langle \sigma_i^z \sigma_j^z \rangle_{K,h}, \quad (37)$$

where $\langle \cdot \rangle_{K,h}$ denotes thermal average with temperature $\beta^{-1} = J/K$ under a transverse field h . Substitution of the above quantity into eq. (26) yields

$$\left[\langle \sigma_i^z \sigma_j^z \rangle_{K,h} \right] = \left[\langle \xi_i \xi_j \rangle_{K_p}^{\text{cl}} \langle \sigma_i^z \sigma_j^z \rangle_{K,h} \right]. \quad (38)$$

Here $\langle \xi_i \xi_j \rangle_{K_p}^{\text{cl}}$ is the correlation function for the classical Ising system with the same configuration $\{\tau_{ij}\}$ and no external field. By taking the absolute value of both sides of this equation, we find

$$\left| \left[\langle \sigma_i^z \sigma_j^z \rangle_{K,h} \right] \right| \leq \left| \left[\langle \xi_i \xi_j \rangle_{K_p}^{\text{cl}} \right] \right|, \quad (39)$$

where we used the fact that correlation function $\langle \sigma_i^z \sigma_j^z \rangle_{K,h}$ does not exceed unity. Similarly, we can prove

$$\left| \left[\text{sgn} \left(\langle \sigma_i^z \sigma_j^z \rangle_{K,h} \right) \right] \right| \leq \left| \left[\langle \xi_i \xi_j \rangle_{K_p}^{\text{cl}} \right] \right| \quad (40)$$

$$\left[\frac{\langle \xi_i \xi_j \rangle_{K_p}^{\text{cl}}}{\langle \sigma_i^z \sigma_j^z \rangle_{K,h}} \right] = \left[\frac{1}{\langle \sigma_i^z \sigma_j^z \rangle_{K,h}} \right] \geq 1. \quad (41)$$

If the probability p of the FM interaction is less than the critical probability p_c at the multicritical point for the classical Ising system, the right-hand side of the inequality (39) vanishes in the limit $|i - j| \rightarrow \infty$. Thus the correlation function for the transverse Ising model on the left-hand side is also equal to zero. Therefore the region of the FM phase is restricted to the range $p > p_c$ (Fig. 1). Since the transverse field represents quantum fluctuations, the correlation function for the transverse Ising model should be reduced from classical system with $h = 0$, which is the physical origin of the above-mentioned restriction.

Next, let us consider how to define a correlation function of the QGG which has convenient properties for the gauge theory. In the CGG, a useful correlation function is defined in terms of an exponential function as $e^{i(\phi_i - \phi_j)}$. This is rewritten using the notation of eq. (12) as

$$\begin{aligned} e^{i(\phi_i - \phi_j)} &= \mathbf{S}_i \cdot \mathbf{S}_j - i(\mathbf{S}_i \times \mathbf{S}_j)_z \\ &= {}^t \mathbf{S}_i \mathbf{S}_j - i {}^t \mathbf{S}_i \mathbf{R}(-\pi/2) \mathbf{S}_j. \end{aligned} \quad (42)$$

This motivates us to define a correlation operator γ_{ij} for the quantum gauge glass as,

$$\gamma_{ij} = (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) - i(\sigma_i^x \sigma_j^y - \sigma_i^y \sigma_j^x). \quad (43)$$

The gauge transformation U changes this operator according to

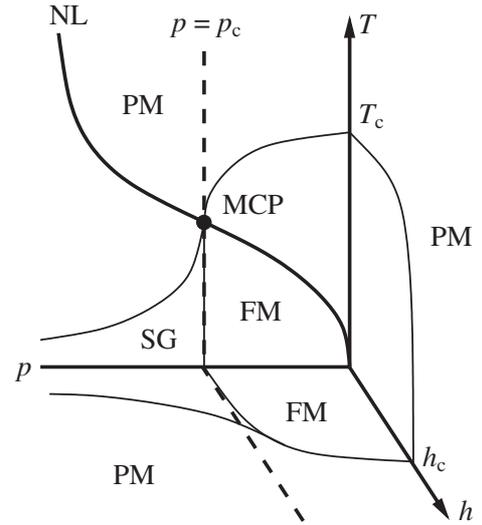


Fig. 1. The phase diagram of the transverse Ising model. The paramagnetic (PM), the ferromagnetic (FM) and spin glass (SG) phases meet at the multicritical point (MCP) in the plane $h = 0$. The dashed line at $p = p_c$ sets a bound for the existence of the FM phase also for $h \neq 0$.

$$U \gamma_{ij} = G \gamma_{ij} G^{-1} = e^{-i(\psi_i - \psi_j)} \gamma_{ij}. \quad (44)$$

The first factor is the same as the correlation function appearing in the classical gauge glass except for a minus sign.

Usually, the correlation function for XY -like systems is defined as the expectation value of $\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y$, which corresponds to $\cos(\phi_i - \phi_j)$ for classical systems. However, this expression is not useful for the gauge theory because it does not separate into gauge variables and spin operators after transformation by U as

$$\begin{aligned} U(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) &= \cos(\psi_i - \psi_j)(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \\ &\quad - \sin(\psi_i - \psi_j)(\sigma_i^x \sigma_j^y - \sigma_i^y \sigma_j^x). \end{aligned} \quad (45)$$

In addition, one can easily prove that

$$\left[\langle \gamma_{ij} \rangle_K \right] = \left[\langle \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \rangle_K \right] \quad (46)$$

since the QGG is invariant under the following transformation:

$$(\sigma_i^x, \sigma_i^y, \sigma_i^z) \rightarrow (\sigma_i^y, -\sigma_i^x, \sigma_i^z), \quad \omega_{ij} \rightarrow -\omega_{ij}. \quad (47)$$

Thus we choose to discuss the correlation operator (43).

To derive an inequality similar to eq. (39), it is useful to consider

$$Q = \gamma_{ij}^\dagger \langle \gamma_{ij} \rangle_K. \quad (48)$$

It is easy to prove that this quantity is gauge invariant because of the property of γ_{ij} given in eq. (44). Substituting this quantity into eq. (34), we obtain

$$\left[\langle \gamma_{ij} \rangle_K \right] = \left[\langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \langle \gamma_{ij} \rangle_K \right], \quad (49)$$

where $\langle \cdot \rangle_{K_p}^{\text{cl}}$ stands for the expectation value for the CGG on the NL. By taking the absolute value of both sides of this equation, we find

$$\left| \left[\langle \gamma_{ij} \rangle_K \right] \right| \leq \left| \left[\langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right] \right|. \quad (50)$$

Similarly, we obtain

$$\left| \left[\text{sgn}(\langle \gamma_{ij} \rangle_K) \right] \right| \leq \left[\left| \langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right| \right], \quad (51)$$

$$\left[\frac{\langle e^{i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}}}{\langle \gamma_{ij} \rangle_K} \right] = \left[\frac{1}{\langle \gamma_{ij} \rangle_K} \right] \geq 1. \quad (52)$$

Since the lower critical dimension d_l is two for continuous spin systems, for $d > 2$, we expect a FM phase to exist at low temperature under small randomness. In this case, a two-spin correlation function tends to the square of magnetization when the two spins are sufficiently separated,

$$\left[\langle \gamma_{ij} \rangle_K \right] \rightarrow m(K, K_p)^2, \quad |i - j| \rightarrow \infty. \quad (53)$$

The right-hand side of inequality (50) is estimated as follows:

$$\begin{aligned} & \left[\left| \langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right| \right]^2 \leq \left[\left| \langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right|^2 \right] \\ & \rightarrow \left[\left| \langle e^{-i\psi_i} \rangle_{K_p}^{\text{cl}} \right|^2 \right] \left[\left| \langle e^{i\psi_j} \rangle_{K_p}^{\text{cl}} \right|^2 \right] \quad (|i - j| \rightarrow \infty) \quad (54) \\ & = q^{\text{cl}}(K_p, K_p)^2 = m^{\text{cl}}(K_p, K_p)^2, \end{aligned}$$

where we used the identity $m^{\text{cl}} = q^{\text{cl}}$ resulting from the gauge theory on the NL for the CGG. Therefore we obtain

$$m(K, K_p)^2 \leq m^{\text{cl}}(K_p, K_p). \quad (55)$$

If the parameter K_p is smaller than the critical point K_{pc}^{cl} for the classical system, the right-hand side vanishes. Consequently, the FM phase for the quantum gauge glass lies in the region satisfying $K_p > K_{pc}^{\text{cl}}$. This result is consistent with the intuitive picture that quantum effects reduce long-range order.

If the spatial dimensionality of the system is equal to the lower critical dimension, $d = 2$, there is no long-range order but quasi long-range order. The Kosterlitz–Thouless (KT) phase²⁵⁾ exists when both K and K_p are sufficiently large. The ordering tendency of the KT phase is observed by the correlation length ξ in the paramagnetic (PM) phase as

$$\left[\langle \gamma_{ij} \rangle_K \right] \sim e^{-|i-j|/\xi_m(K, K_p)}, \quad (56)$$

$$\begin{aligned} & \left[\left| \langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right|^2 \right] \\ & = \left[\langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \right] \sim e^{-|i-j|/\xi_m^{\text{cl}}(K_p, K_p)}. \quad (57) \end{aligned}$$

From the square of the inequality (50), the limit $|i - j| \rightarrow \infty$ yields

$$\xi_m(K, K_p) \leq 2\xi_m^{\text{cl}}(K_p, K_p). \quad (58)$$

Thus, the KT phase for the QGG is also restricted to the region $K_p > K_{pc}^{\text{cl}}$.

5. Ground State Property of Quantum Gauge Glass

Next we consider the ground state property of the QGG. It is necessary to consider the transformation rule for eigenstates of the Hamiltonian. Let us denote an eigenstate of the Hamiltonian with the eigenvalue x by $|x; \omega\rangle$,

$$H|x; \omega\rangle = x|x; \omega\rangle. \quad (59)$$

Since the invariance of the Hamiltonian (23) can be rewritten as

$$G(VH)G^{-1} = H, \quad (60)$$

we have an eigen-value equation

$$(VH)G^{-1}|x; \omega\rangle = xG^{-1}|x; \omega\rangle. \quad (61)$$

Note that the effect of the operator V is restricted to the inside of the brackets (\dots) . The state $G^{-1}|x; \omega\rangle$ is an eigenstate of the gauge-transformed Hamiltonian (VH) . Therefore, one can derive the transformation rule of the eigenstate

$$V|x; \omega\rangle = G^{-1}|x; \omega\rangle. \quad (62)$$

This rule is derived straightforwardly, if the state is not degenerate. When the state is degenerate, the rule is not unique. However, it is reasonable and has no problem, if we use this rule as the transformation rule.

Now, we consider the ground state. Let us denote the ground state of the Hamiltonian by $|g; \omega\rangle$. The above transformation rule leads to the gauge transformation of the average of any operator Q in the ground state;

$$V\langle g; \omega|Q|g; \omega\rangle = \langle g; \omega|G(VQ)G^{-1}|g; \omega\rangle. \quad (63)$$

The thermal average $\langle \dots \rangle_K$ in equations derived in the previous section can be replaced by the ground state expectation value. For example,

$$\left[\langle g; \omega|\gamma_{ij}|g; \omega\rangle \right] = \left[\langle e^{-i(\psi_i - \psi_j)} \rangle_{K_p}^{\text{cl}} \langle g; \omega|\gamma_{ij}|g; \omega\rangle \right] \quad (64)$$

is derived instead of eq. (49), which provides the inequality for the order parameters in the ground state,

$$m(\infty, K_p)^2 \leq m^{\text{cl}}(K_p, K_p) \quad (65)$$

instead of eq. (55), and

$$\xi_m(\infty, K_p) \leq 2\xi_m^{\text{cl}}(K_p, K_p) \quad (66)$$

instead of eq. (58).

In two dimensions, it has been shown that the FM long range order exists in the ground state of the pure quantum XY model ($K_p = +\infty$).²⁶⁾ However, in the disordered regime ($K_p < +\infty$), the FM order must disappear since the FM order in the CGG model, the right-hand side of the inequality (65), does not exist. The only possibility in this regime is the existence of the KT phase, which is consistent with the inequality (66).

6. Phase Diagram of Quantum Mattis Model

In this section, we introduce and discuss the properties of the quantum XY Mattis model which has no frustration. Using the gauge transformation, one can obtain the phase diagram for this model explicitly. One of the phase boundaries is determined by the critical point of the pure quantum system and the other by that of the classical one. This is an important difference from the classical non-frustrated systems.²⁷⁾

Let us locate a quenched random variable ω_i at each site and define the phase factor ω_{ij} as $\omega_{ij} = \omega_j - \omega_i$. The Hamiltonian for the quantum Mattis model is defined in terms of the pure quantum XY model, H_0 , that is,

$$H = G_\omega H_0 G_\omega^{-1}, \quad G_\omega = \prod_i \exp\left(-\frac{i\omega_i}{2} \sigma_i^z\right). \quad (67)$$

Thus, the ground-state energy is always the same as that of H_0 and the ground state is obtained by operating G_ω on the ground state of the pure system. This system has no frustration in this sense. It is easy to show that the Hamiltonian for this model is invariant under the gauge transformation, $(UV)H = H$, where the gauge transformation for the configuration is defined as $\omega_i \rightarrow \omega_i - \psi_i$.

Using eqs. (44) and (67), we immediately obtain

$$\langle \gamma_{ij} \rangle_K = e^{i(\omega_i - \omega_j)} \langle \gamma_{ij} \rangle_{0,K}, \quad (68)$$

where the angular brackets on the right-hand side denote the thermal average with respect to H_0 . Note that the correlation function for the pure system $\langle \gamma_{ij} \rangle_{0,K}$ does not depend the quenched variable ω_i . Here, we assume that the distribution function for the quenched random variable is proportional to the Boltzmann factor of the pure classical XY model,

$$P(\omega) = \frac{e^{K_p \sum_{ij} \cos(\omega_i - \omega_j)}}{Z_0^{\text{cl}}(K_p)}. \quad (69)$$

Hereafter, the subscript 0 and the superscript cl stand for pure and classical systems, respectively. Then the configuration average is equal to the thermal average for the pure classical XY model with coupling K_p . Consequently, we find

$$[\langle \gamma_{ij} \rangle_K] = \langle e^{i(\omega_i - \omega_j)} \rangle_{0,K_p}^{\text{cl}} \langle \gamma_{ij} \rangle_{0,K}. \quad (70)$$

Similarly, from eq. (68), the spin-glass correlation function satisfies

$$\left[\left| \langle \gamma_{ij} \rangle_K \right|^2 \right] = \left| \langle \gamma_{ij} \rangle_{0,K} \right|^2. \quad (71)$$

Taking the limit $|i - j| \rightarrow \infty$ of eqs. (70) and (71), we obtain

$$m(K, K_p) = m_0^{\text{cl}}(K_p) m_0(K), \quad (72)$$

$$q(K, K_p) = m_0(K)^2. \quad (73)$$

For $d > 2$, there are three phases, PM, FM and Mattis spin-glass (MSG) phases. Figure 2(a) shows the phase diagram. The phase boundary between PM and other phases is at $K = K_{0c}$ and the one between FM and MSG is at $K = K_{0c}^{\text{cl}}$.

If $d = 2$, the correlation length determines the phase structure. From eqs. (70) and (71), we find

$$\frac{1}{\xi_m(K, K_p)} = \frac{1}{\xi_0^{\text{cl}}(K_p)} + \frac{1}{\xi_0(K)} \quad (74)$$

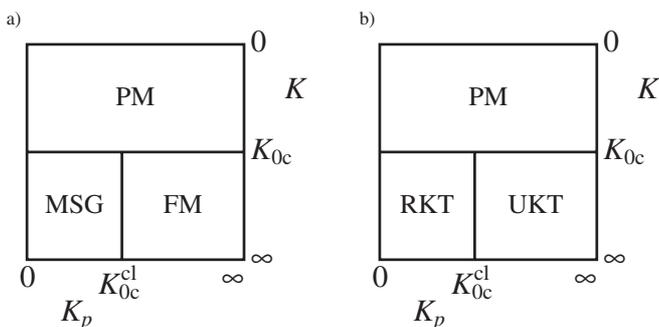


Fig. 2. (a) The phase diagram for the quantum XY Mattis model in $d > 2$. There are three kinds of phases, the PM, the FM and the SG. (b) The phase diagram in $d = 2$. There exist three kinds of phases, the PM, the uniform KT (UKT) and the random KT (RKT).

$$\xi_q(K, K_p) = \frac{\xi_0(K)}{2}, \quad (75)$$

where $\xi_q(K, K_p)$ denotes the spin-glass correlation length. Thus, similarly to the $d > 2$ case, three phases exist: (i) $\xi_m < \infty$ and $\xi_q < \infty$: paramagnetic phase (PM), (ii) $\xi_m = \infty$ and $\xi_q = \infty$: uniform KT phase (UKT) and (iii) $\xi_m < \infty$ and $\xi_q = \infty$: random KT phase (RKT). The phase diagram is shown in Fig. 2(b).

We note again that the location of the horizontal phase boundary is determined by the critical point of the quantum pure system, K_{0c} , and the vertical one comes from that of the classical pure system, K_{0c}^{cl} .

7. Conclusions

In this paper, we have investigated quantum spin glasses using the gauge theory. First, we considered the transverse Ising model and the QGG. To construct the gauge theory, we defined the gauge transformation by rotational operator on the Hilbert space. It is essential that interaction of these models is written in term of one or two components of spin operators. If a system has Heisenberg-type interactions, we can not define a gauge transformation which satisfies the commutation rule.

Using the gauge theory, we obtained mainly two results. One is the identity for gauge invariant operators. The FM limit state and the classical equilibrium state on the NL provide the same expectation value for gauge invariant operators. We note that this result remains valid when we introduce time evolution following the Schrödinger equation. This result has already been pointed out for classical spin glasses with stochastic dynamics.^{23,24} We have shown that the same applies to quantum spin systems.

The other result is a set of inequalities for the correlation function. These inequalities show that the correlation function for the quantum model never exceeds the classical counterpart on the NL. The corresponding classical system is determined by a transformation rule for probability distribution. As a result, the order parameter is smaller than the square of the classical one. Therefore the FM phase (or the KT phase) should lie within the corresponding classical one. This is natural intuitively since quantum effects reduce ordering tendency, but to prove it rigorously is a different and quite a non-trivial problem. Moreover, these results are valid even if the system is in the ground state. Thus FM long range order vanishes in the two-dimensional QGG although the ground state of the pure quantum XY model has FM order.

Next, we determined the phase diagram for the quantum XY Mattis model. This model is not a real SG because of lack of frustration. It is interesting, nevertheless, that both quantum and classical phase transitions occur in a single system, which may serve as a starting point for investigations of more realistic quantum spin glasses with frustration.

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